equation in the two independent variables, $\psi_1$ and $\psi_2$. In addition, by forming the functions

$$\frac{\Delta V}{V_{\infty 1}} = \left(\frac{a_1 \Delta V^2}{\mu}\right)^{1/2} \qquad \frac{r V_{\infty 1}^2}{\mu} = \frac{r}{a_1} \tag{A9}$$

in Eq. (A6) the entire solution can be reduced to two input parameters: $a_2/a_1$ [or $(V_{\infty 1}/V_{\infty 2})^2$] and $\kappa$. The optimization is thus made independent of the planet and of the initial planetocentric velocity $V_{\infty 1}$.

## Appendix B: Periapsis Transfer

If it is required that the impulse be applied tangentially at the periapsis, the equations of Appendix A are greatly simplified. When $\eta_1 = \eta_2 = 0$, it can be shown that

$$\frac{r}{a_1} = \frac{(a_2/a_1)(\sin\psi_2/\sin\psi_1)(\tan^2\psi_1/\tan^2\psi_2) - 1}{\tan^2\psi_1[(\sin\psi_2/\sin\psi_1) - 1]} \tag{B1}$$

$$\kappa = \psi_1 + \psi_2 \tag{B2}$$

$$\sin\psi_1\left(\frac{1}{(r/a_1)\tan^2\psi_1} - 1\right) = 1 \tag{B3}$$

Eliminating $r/a_1$ from Eqs. (B1) and (B2) produces an expression relating $\psi_1$ and $\psi_2$:

$$\sin\psi_1 = \frac{\sin\psi_2}{(a_2/a_1) + [1 - (a_2/a_1)]\sin\psi_2} \tag{B4}$$

Finally, the requirement that the impulse be tangential is expressed by

$$\cos\theta_1 = \cos\theta_2 \tag{B5}$$

and the equation for $\Delta V/V_{\infty 1}$ becomes

$$\frac{\Delta V}{V_{\infty 1}} = \left(1 + \frac{a_1}{a_2} + \frac{2}{r/a_1} \times \left[2 - \left\{\left(2 + \frac{r}{a_1}\right)\left(2 + \frac{a_1}{a_2}\frac{r}{a_1}\right)\right\}^{1/2}\right]\right)^{1/2} \tag{B6}$$

From Eqs. (B2) and (B4), the constraining equation is

$$\sin(\kappa - \psi_2) = \frac{\sin\psi_2}{(a_2/a_1) + [1 - (a_2/a_1)]\sin\psi_2} \tag{B7}$$

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# Derivation of Nodal Period of an Earth Satellite and Comparisons of Several First-Order Secular Oblateness Results


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**An expression for nodal period is derived to first order in the perturbed gravitational potential of an oblate earth spheroid, i.e., to order $J$ and for small eccentricities $e = O(J_2)$. First-order secular effects on satellite motion around an oblate earth, as obtained by various authors, have been compared and found in agreement. However, some disagreement exists in the expressions obtained for the anomalistic and nodal period of satellite motion by various authors. A consistent set of expressions for these periods is recommended for use in orbital calculations.**


## Nomenclature

$a$ = semimajor axis
$a_0$ = semimajor axis of a Keplerian orbit; unperturbed semimajor axis
$\bar{a}$ = mean semimajor axis of a perturbed orbit
$b$ = constant defined by Eq. (8)
$e$ = eccentricity, with a varying interpretation by different authors
$e_0$ = eccentricity constant
$f$ = earth's flattening $\simeq 1/298.25$
$h$ = satellite altitude
$\bar{h}$ = average satellite altitude in the region $0° \leq \beta < 360°$
$i_0$ = average orbit inclination about which $i$ varies periodically; $i_0$ is measured positive in counterclockwise direction from due east on the equator to orbital plane at the ascending node, so that $0° \leq i_0 \leq 180°$
$J_2$ = second zonal harmonic coefficient in the expansion of earth's gravitational potential $1.08228 \times 10^{-3}$
$k$ = constant defined by Eq. (3)
$n_0$ = mean motion of an unperturbed satellite orbit
$n_r$ = perturbed anomalistic mean motion of a satellite

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$n_\beta$ = perturbed nodal mean motion of a satellite
$n_\rho$ = mean motion in the oblate spheroidal coordinate $\rho$
$p$ = angular momentum per unit mass about the polar axis, a constant of the orbit
$P$ = semilatus rectum of a Keplerian orbit
$\bar{P}$ = mean semilatus rectum of a perturbed orbit, $\bar{P} = \bar{a}(1 - e_0^2)$
$r$ = instantaneous orbital radius
$r_0$ = orbital radius constant of a Keplerian orbit, equivalent to $P$
$\bar{r}_0$ = perturbed orbital radius constant, which is related to $p$
$R_e$ = equatorial radius of the earth, 3443.927 naut miles
$t$ = time
$t_{orb}$ = time in orbit
$\beta$ = instantaneous central angle measured from mean line of nodes; measured positive in direction of satellite motion
$\mu$ = gravitational constant of earth, $1.407645 \times 10^{16}$ ft$^3$/sec$^2$
$\rho$ = oblate spheroidal coordinate
$\Delta\tau$ = time for satellite to traverse the angle $(d\omega/dt)\Delta\tau$
$\tau_r$ = anomalistic period of satellite
$\tau_\beta$ = nodal period of satellite
$\omega$ = angle of perigee, i.e., angle from ascending node to perigee
$d\omega/dt$ = secular rotational rate of the perigee
$\Omega$ = instantaneous position of the line of nodes
$d\Omega/dt$ = secular regression rate of the mean line of nodes

## Introduction

THE closed orbit of a satellite about a spherically symmetrical central body, and having no perturbing forces acting on it, is a fixed circle or ellipse in a fixed plane. The period of motion is the same, regardless of the reference chosen. This type of motion will be known as unperturbed or Keplerian satellite motion. Oblateness of the central body tends to make a twisted space curve out of the satellite orbit.

The analytical methods for determining the effect of oblateness perturbations can be classified as follows: 1) integrate the equations of motion directly by an analytical expansion, and 2) map the satellite orbit, by either of two available methods, as a plane curve on the orbital plane which, at any instant, contains the satellite radius and velocity vectors. In this orbital plane, either approximate the trajectory by an osculating ellipse and obtain results by the method of variation of parameters (astronomical approach), or try to assume the actual equation of the plane curve, which represents a succession of satellite positions, to approach the desired accuracy (Struble).

The satellite trajectory in this orbital plane is very nearly a circle or an ellipse, because of the small oblateness of the earth. To first order in oblateness (using terms to order $J$ in the expansion of the gravitational potential, as given by Jeffreys[1]), the orbital plane precesses around the polar axis at a variable rate (nodal regression). In addition, the line of apsides defining the orientation of the satellite trajectory in the orbital plane rotates in this plane at a variable rate (apsidal rotation). The mean motion, or the average rate of rotation of the satellite in the orbital plane, now depends on the orbital inclination that results in an oblateness correction to the period of the unperturbed motion.

There now are different values for the period of satellite motion, depending on what reference is chosen for the determination of the period. The variables, other than $\Omega, \omega$, and $\tau$, appearing in the description of satellite motion may depend on the orbital inclination or, at most, undergo periodic variations to first order in oblateness. This motion of the satellite in the potential field of an oblate central body will be known as perturbed motion.

The general problem of perturbed satellite motion about an oblate earth has been studied extensively. It has been shown[2] that, at least for small eccentricities $e = O(J_2)$, the first-order periodic variations in satellite radius obtained by the various authors are in agreement. A number of them (Refs. 3–15) have obtained expressions for either the nodal period, the time from one ascending node to the next, or the anomalistic period, the time from one perigee to the next.

In this work, the first-order theory of Struble[16, 17] was used to derive an expression for the nodal period for the case of small eccentricity. The nodal period is more advantageous than the anomalistic or sidereal periods for use in computations, particularly in the case of near-circular or circular orbits, where the position of the perigee is difficult to define. The expressions for 1) the motion of the line of nodes, 2) apsidal motion, and 3) the anomalistic and nodal periods, as obtained by the various authors, are compared, and a consistent set of expressions is recommended for use.

## Derivation of the Nodal Period, $\tau_\beta$

According to the first-order theory of Struble,[16] the variation of orbital central angle $\beta$ with time $t$ is given by

$$dt/d\beta = (k/p)r^2 \tag{1}$$

where for small eccentricities $e = O(J_2)$ the radius $r$ can be written

$$r = \bar{r}_0\,[1 - e\cos(\beta - \omega) + (J_2/4)\,(R_e/\bar{r}_0)^2 \sin^2\bar{i}_0 \cos 2\beta] \tag{2}$$

$$k = \cos\bar{i}_0[1 - \tfrac{3}{2}J_2(R_e/\bar{r}_0)^2 \cos^2\bar{i}_0] = \text{const} \tag{3}$$

$\omega$ is the angle from the ascending node to perigee, $\bar{i}_0$ is the mean orbital inclination, $0° \leq \bar{i}_0 \leq 180°$, $R_e$ is the equatorial radius of the earth, and $p$ is a constant (the angular momentum per unit mass about the polar axis), which is related to the radius constant $\bar{r}_0$ as follows:

$$\frac{1}{\bar{r}_0} = \frac{\mu}{p^2}\cos^2\bar{i}_0 + \frac{3}{4}\frac{J_2}{\bar{r}_0}\left(\frac{R_e}{\bar{r}_0}\right)^2\left(1 + \frac{e^2}{2}\right)(2 - 3\sin^2\bar{i}_0) \tag{4}$$

Squaring Eq. (2) and neglecting terms of the order of $J^2$, then

$$r^2 = \bar{r}_0^2[1 - 2e\cos(\beta - \omega) + (J_2/2)(R_e/\bar{r}_0)^2 \sin^2\bar{i}_0 \cos 2\beta] \tag{5}$$

Since the term $1/p$ appears in Eq. (1), then solving for $1/p$ from Eq. (4), one can obtain

$$\frac{1}{p} = \frac{1}{(\mu\bar{r}_0)^{1/2}\cos\bar{i}_0}\left[1 - \frac{3}{4}J_2\left(\frac{R_e}{\bar{r}_0}\right)^2(2 - 3\sin^2\bar{i}_0)\right]^{1/2} \tag{6}$$

In addition, $d\omega = b\,d\beta$, where

$$b = \tfrac{3}{4}J_2(R_e/\bar{r}_0)^2(5\cos^2\bar{i}_0 - 1) \tag{7}$$

Then, to find $\omega$ explicitly as a function of $\beta$, take

$$\int_{\omega_0}^{\omega} d\omega = b\int_0^{\beta} d\beta \tag{8}$$

from which

$$\omega - \omega_0 = b\beta \text{ or } \omega = b\beta + \omega_0 \tag{9}$$

where $\omega_0$ is the initial value of $\omega$ at $\beta = 0$.

Substitute 1) the expression for $\omega$ from Eq. (9) into Eq. (5) and 2) the expression for $r^2$ from Eq. (5) into Eq. (1). Then, integrating Eq. (1) from $\beta = 0$ to $2\pi$ and from $t = 0$ to $\tau_\beta$, to get the nodal period $\tau_\beta$,

$$\int_0^{\tau_\beta} dt = \frac{k\bar{r}_0^2}{p}\int_0^{2\pi}\left\{1 - 2e\cos[\beta(1 - b) - \omega_0] + \frac{1}{2}J_2\left(\frac{R_e}{\bar{r}_0}\right)^2 \sin^2\bar{i}_0\cos 2\beta\right\}d\beta \tag{10}$$

The integral of the last term is zero, so that after integration

$$\tau_\beta = \frac{k\bar{r}_0^2}{p}\left\{2\pi + 2e\frac{\cos\omega_0}{1-b}[\sin 2\pi b] + 2e\frac{\sin\omega_0}{1-b}[\cos 2\pi b - 1]\right\}$$

Since $2\pi b = d\omega/dt$ rad/rev, then

$$\tau_\beta = \frac{k\bar{r}_0^2}{p}\left\{2\pi + \frac{2e}{1-b}\left[\sin\left(\omega_0 + \frac{d\omega}{dt}\right) - \sin\omega_0\right]\right\} \quad (11a)$$

or

$$\tau_\beta = \frac{k\bar{r}_0^2}{p}\left\{2\pi + \frac{2e}{1-b}\Delta(\sin\omega_0)\right\} \quad (11b)$$

Now the maximum $d\omega/dt \simeq 0.3$ deg/rev, and the maximum $\Delta(\sin\omega_0)$ occurs at $\omega_0 = 0$. In this case,

$$\max\Delta(\sin\omega_0) = \sin(\omega_0 + d\omega/dt) - \sin\omega_0 = 0.004 = O(J) \quad (12)$$

Hence, the second term in Eqs. (11) becomes of the order of $O(J_2^2)$ for small $e$, and can be neglected to first order, so that

$$\tau_\beta = 2\pi k\bar{r}_0^2/p \quad (13)$$

Now, substituting for $p$ from Eq. (6) and for $k$ from Eq. (3) and dropping terms of $O(J^2)$, then, for $e = O(J)$,

$$\tau_\beta = \frac{2\pi}{(\mu)^{1/2}}\bar{r}_0^{3/2}\left\{1 - \frac{3}{8}J_2\left(\frac{R_e}{\bar{r}_0}\right)^2[7\cos^2 i_0 - 1]\right\} \quad (14)$$

which agrees with the nodal period given by both Sterne[9] and King-Hele[5] for this case.

It is interesting to note that, for small eccentricities, $\bar{r}_0$ has the physical significance of being the average orbital radius about which $r$ varies periodically, as can be seen from Eq. (2). In addition, the forementioned expression for nodal period is valid for circular as well as for near-circular orbits.

## Discussion of First-Order Secular Oblateness Results

The secular change of satellite orbital characteristics due to earth oblateness is of major interest, since it is often desired (for mission planning) to obtain the average satellite behavior over a long period of time. In the following, several first-order secular oblateness results will be discussed in detail.

The mean rate of nodal regression to first order has been found by many authors: Kozai,[3] Brouwer,[4] Groves,[13] King-Hele,[5] Blitzer,[11] Vinti,[6] Anthony,[18] Jaramillo,[12] and others. In units of degrees per nodal period (time between successive ascending nodes), it is

$$\frac{d\Omega}{dt} = -540J_2\frac{R_e^2}{a^2(1-e^2)^2}\cos i_0 \quad \frac{\text{deg}}{\text{nodal period}} \quad (15)$$

Since $d\Omega/dt = O(Jd\beta/dt)$, any one of the following expressions can be used to first order in the denominator of Eq. (15) and also in Eqs. (16–18):

$$a^2(1-e^2)^2,\ a_0^2(1-e^2)^2,\ \bar{a}^2(1-e^2)^2,\ P^2,\ \bar{P}^2,\ r_0^2,\ \bar{r}_0^2$$

The value of $d\Omega/dt$ is negative for a direct orbit ($i_0 < 90°$), i.e., it is a shift toward the west, whereas $d\Omega/dt$ is positive for a retrograde orbit ($i_0 > 90°$). The nodal period is chosen here because the orbital central angle $\beta$, measured from the ascending node, increases by 360° in one nodal period, and the ascending node is an easily identifiable point in all orbits with the exception of the equatorial where it can be fixed arbitrarily and the regression rate obtained by a limiting process as $i \to 0°$ or 180°.

Struble and Campbell[17] obtain the instantaneous line of nodes to first order. By replacing their variable $r_0$ by $a(1-e^2)$ [cf., Eqs. (17) and (25), Ref. 16], one obtains

$$\Omega = -\frac{3}{2}J_2\frac{R_e^2}{a^2(1-e^2)^2}\cos i_0\left[\beta - \frac{1}{2}\sin 2\beta + e\sin(\beta-\omega) - \frac{1}{6}e\sin(3\beta-\omega) - \frac{1}{2}e\sin(\beta+\omega)\right] \quad (16)$$

If one considers the change in $\Omega$ during one nodal period, the mean regression rate given by Eq. (15) can be derived from Eq. (16).

Similarly, the mean rate of apsidal rotation to first order has been found by many authors (Kozai,[3] Brouwer,[4] Groves,[13] King-Hele,[5] Blitzer,[11] Vinti,[6] Anthony,[18] Jaramillo,[12] and others) in units of degrees per nodal period:

$$\frac{d\omega}{dt} = 540J_2\frac{R_e^2}{a^2(1-e^2)^2}\left(\frac{5\cos^2 i_0 - 1}{2}\right) \quad \frac{\text{deg}}{\text{nodal period}} \quad (17)$$

For near-polar orbits the perigee regresses, i.e., it moves against the direction of satellite motion, whereas for near-equatorial orbits the perigee advances, i.e., it moves in the direction of satellite motion. The inclination angles at which $5\cos^2 i_0 - 1 = 0$, or $i_0 \simeq 63.45°$ and $i_0 \simeq 116.55°$ are known as critical inclination angles. Near the critical inclination, when $5\cos^2 i_0 - 1 = O(J_2)$, a first-order theory is insufficient to describe the apsidal motion, so that Eq. (17) does not hold. Garfinkel,[19–21] Hori,[22] Hagihara and Kozai,[23] and Struble[16] have found that near the critical inclination the apsidal motion consists of an oscillation about the points of maximum latitude in either hemisphere.

It should be kept in mind that for near-circular orbits, when $e = O(J_2)$, the geocentric radius $r$ at the central angle $\beta = \omega$ is not necessarily a minimum. For circular orbits ($e = 0$), the perturbed satellite geocentric radius describes a fixed ellipse about the oblate earth, with center at the center of the earth, minima at the points of maximum latitude and maxima at the nodes (see also Izsak[24]). One can still define an arbitrary initial $\omega$ for purposes of calculation and give the apsidal rotation by Eq. (17) in the limit for the circular case, although the minimal $r$ occurs at fixed points in the orbit.

Struble and Campbell,[17] with a slight change of notation, give

$$\frac{d\omega}{d\beta} = \frac{3}{2}J_2\frac{R_e^2}{a^2(1-e^2)^2}\left(\frac{5\cos^2 i_0 - 1}{2}\right) \quad (18)$$

which can be shown to reduce to Eq. (17).

Now turn to the various periods used in the description of satellite motion. Kepler's third law relates the mean motion of the unperturbed satellite orbit, $n_0$[rad/sec], to the unperturbed semimajor axis $a_0$,

$$n_0^2 a_0^3 = \mu \quad (19)$$

where $\mu$ is the gravitational constant of the central body. The period associated with unperturbed satellite motion is $\tau_0 = 2\pi/n_0$.

In perturbed satellite motion or when first-order perturbing forces due to earth oblateness act on the satellite, one can define three, in general distinct, periods of motion:

1) The anomalistic period, $\tau_r$, the time from one perigee to the next. In that time the elliptic angles (true, mean, and eccentric anomaly) increase by 360°, whereas the central angle $\beta$ increases by more or less than 360°, depending on whether the apsidal rotation is against or in the direction of satellite motion.

2) The nodal period, $\tau_\beta$, also called draconic period, the time from one ascending node to the next. In that time, the central angle $\beta$ increases by 360°, since $\beta$ is measured from the instantaneous position of the ascending node.

3) The sidereal period, the time between successive closest approaches of the satellite to the same point in inertial space.

In artificial satellite theory, the sidereal period is less important than the other two periods. It is rarely used and will not be discussed any further here.

In many satellite calculations, especially in approximate ephemeris prediction, it is more convenient to use the nodal period than the anomalistic period. Unfortunately, however, differing expressions have been obtained for both in the literature. Various expressions for the periods of perturbed satellite motion will now be discussed.

### Kozai,[3] Brouwer,[4] and Groves[13]

These authors use the astronomical approach. It is convenient to introduce a mean semimajor axis $\bar{a}$ for perturbed satellite motion (Kozai):

$$\bar{a} = a_0 \left\{1 - \frac{\frac{3}{2}J_2R_e^2}{a_0^2(1-e^2)^2}\left(\frac{3\cos^2 i_0 - 1}{2}\right)(1-e^2)^{1/2}\right\} \quad (20)$$

The perturbed anomalistic mean motion to first order can be given in terms of $n_0$ (Kozai, Brouwer, Groves) as

$$n_r = n_0 \left\{1 + \frac{\frac{3}{2}J_2R_e^2}{\bar{a}^2(1-e^2)^2}\left(\frac{3\cos^2 i_0 - 1}{2}\right)(1-e^2)^{1/2}\right\} \quad (21)$$

where $\bar{a}$ and $n_r$ satisfy the following defining relation for $\bar{a}$:

$$n_r^2\bar{a}^3 = \mu\left\{1 - \frac{\frac{3}{2}J_2R_e^2}{\bar{a}^2(1-e^2)^{3/2}}\,\frac{3\cos^2 i_0 - 1}{2}\right\} \quad (22)$$

If one compares Eq. (22) with Eq. (19), then the "effective gravitational constant" of the central body, as given on the right-hand side of Eq. (22), varies with the orbital inclination, due to the oblateness of the central body.

The anomalistic period can be obtained directly from Eq. (22):

$$\tau_r = \frac{2\pi}{n_r} = \frac{2\pi}{(\mu)^{1/2}}(\bar{a})^{3/2}\left\{1 + \frac{\frac{3}{2}J_2R_e^2}{\bar{a}^2(1-e^2)^{3/2}}\left(\frac{3\cos^2 i_0 - 1}{4}\right)\right\} \quad (23)$$

At inclination angles of $i_0 \simeq 54.7$ deg and $i_0 \simeq 125.3$ deg, $3\cos^2 i_0 = 1$ and $\tau_r = \tau_0$. Physically, this is due to a combination of the assumed mass distribution of the earth and the apsidal rotation at these inclination angles.

### Struble[16,17]

Struble's unperturbed $r_0$ is the semilatus rectum $P = a(1 - e^2)$ of a Keplerian orbit, as can be seen by direct comparison of Eqs. (17) and (27) in Ref. 16. Its perturbed value, $\bar{r}_0$, is given in Eq. (28) of the same reference. It is defined in terms of the angular momentum about the polar axis by Eq. (4), and it can be rewritten as

$$\bar{r}_0 = r_0\left[1 - \frac{3}{2}J_2\left(\frac{R_e}{\bar{r}_0}\right)^2\frac{3\cos^2 i_0 - 1}{2}\left(1 + \frac{e^2}{2}\right)\right] \quad (24)$$

For circular orbits ($e = 0$), one finds from Eqs. (20) and (24), that $\bar{r}_0 = \bar{a} = \bar{P}$, where $\bar{P} = \bar{a}(1 - e^2)$ is the perturbed semilatus rectum, and Struble's results agree with those of Kozai. This holds true also for near-circular orbits, so long as $e = O(J_2)$. This cannot be said for larger eccentricities, since the meaning of $e$ is not the same in the astronomical approach, where it is the variable eccentricity of the osculating ellipse or an eccentricity constant, as it is in Struble's approach, where $e$ is a constant to first order in $J$ describing the curve traced by the satellite in its orbital plane. The nodal period in terms of $\bar{r}_0$ for $e = O(J_2)$ was obtained in Eq. (14).

From the first two forementioned groups of authors and from Keplerian motion, it is evident that the variables $\bar{a}$, $n_r$, $\bar{r}_0$, $\tau_r$, and $\tau_\beta$ depend on the orbital inclination $i_0$, whereas the unperturbed values of $a_0$, $n_0$, $r_0$, and $\tau_0$ are independent of $i_0$.

### Sterne[8,9] and Garfinkel[19,20]

Sterne, Garfinkel, and Vinti use the astronomical approach and define an intermediary orbit, i.e., the orbit of a satellite in a potential field other than, but similar to, the one for the earth. For this intermediary orbit, which closely approximates the perturbed orbit, Sterne obtains expressions for the anomalistic and nodal mean motions in Ref. 9. The periods corresponding to these mean motions are

$$\tau_r = \frac{2\pi}{n_r} \simeq \frac{2\pi}{(\mu)^{1/2}}(\bar{a})^{3/2}\left\{1 + \frac{3}{2}J_2\frac{R_e^2}{\bar{a}^2(1-e^2)^{3/2}} \times \left[\frac{3\cos^2 i_0 - 1}{4} - (1 - (1-e^2)^{1/2})\frac{3\cos^2 i_0 - 1}{4}\right]\right\} \quad (25)$$

$$\tau_\beta = \frac{2\pi}{n_\beta} \simeq \frac{2\pi}{(\mu)^{1/2}}(\bar{a})^{3/2}\left\{1 - \frac{3}{2}J_2\frac{R_e^2}{\bar{a}^2(1-e^2)^2} \times \left[\frac{7\cos^2 i_0 - 1}{4} + \frac{e^2}{4}(3\cos^2 i_0 - 1)\right]\right\} \quad (26)$$

Comparison of Eqs. (23) and (25) shows that Sterne has an additional term

$$-\frac{3}{2}J_2\frac{R_e^2}{\bar{a}^2(1-e^2)^{3/2}}\,\frac{3\cos^2 i_0 - 1}{4}\,[1 - (1-e^2)^{1/2}] \quad (27)$$

in his expression for the anomalistic period. This term vanishes for $e = 0$, it is nearly zero for small eccentricities, and, even for $e = 0.1$, it is only 0.5% of Kozai's perturbation term:

$$J\frac{R_e^2}{\bar{a}^2(1-e^2)^{3/2}}\,\frac{3\cos^2 i_0 - 1}{4} \quad (28)$$

Hence, Eq. (27) may be considered of second order for $e \leq 0.1$. Because of the assumptions and techniques employed by Kozai, Brouwer, and Groves, the form Eq. (23) is to be preferred for the anomalistic period.

The nodal periods, Eqs. (14) and (26), can also be compared to first order so long as $e \leq (J_2)^{1/2}$ and so long as the different definitions that Struble and Sterne, as well as Kozai, Brouwer, and Groves give for $e$ are kept in mind.

### Vinti[6,7]

Vinti introduces a gravitational potential for his intermediary orbit, which accounts for all the second-zonal harmonic (first order in oblateness) and more than half of the fourth-zonal harmonic in the earth's gravitational potential. His results in the mean motion in the oblate spheroidal coordinate $\rho$ and in the nodal mean motion, with a slight change in notation, are

$$n_\rho = n + O(J_2^2) \quad (29)$$

$$n_\beta = n\left\{1 + \frac{\frac{3}{2}J_2R_e^2}{\bar{a}^2(1-e^2)^2}\left(\frac{10\cos^2 i_0 - 2}{4}\right)\right\} \quad (30)$$

Vinti obtains the result (Ref. 7, p. 190) that the $\rho$ perigee, in oblate spheroidal coordinates, and the $r$ perigee, in spherical coordinates, rotate with the same rate relative to the ascending node (apsidal rotation) to first order in $J$. Hence, it would seem appropriate to use $n_r$, as obtainable from Eq. (22), for $n$ in Eqs. (29) and (30). With this reinterpretation of $n$, the anomalistic period is given by Eq. (23), and the nodal period can be written to first order, as

$$\tau_\beta = \frac{2\pi}{n_\beta} = \frac{2\pi\bar{a}^{3/2}}{u^{1/2}}\left\{1 - \frac{\frac{3}{2}J_2R_e^2}{a^2(1-e^2)^2} \times \left[\frac{10\cos^2 i_0 - 2}{4} - \frac{3\cos^2 i_0 - 1}{4}(1-e^2)^{1/2}\right]\right\} \quad (31)$$

For small $e$ the nodal period becomes

$$\tau_\beta = \frac{2\pi}{(\mu)^{1/2}}\bar{a}^{3/2}\left\{1 - \frac{\frac{3}{2}J_2R_e^2}{\bar{a}^2(1-e^2)^2}\,\frac{7\cos^2 i_0 - 1}{4}\right\} \quad (32)$$

**King-Hele[5,14]**

By using a method similar to Struble's, King-Hele [Ref. 5, Eq. (73)] obtains the following first-order expression for the period in terms of the mean semimajor axis $\bar{a}$, if the eccentricity $e \leq (J)^{1/2}$:

$$\tau_\beta = \frac{2\pi}{(\mu)^{1/2}} (\bar{a})^{3/2} \left\{1 - \frac{\frac{3}{2}J_2 R_e^2}{\bar{a}^2} \frac{7 \cos^2 i_0 - 1}{4}\right\} \quad (33)$$

In terms of the perturbed mean semilatus rectum $\bar{P} = \bar{a}(1 - e^2)$, King-Hele obtains the following expression for the period [Ref. 5, Eq. (71)]:

$$\tau_\beta = \frac{2\pi}{(\mu)^{1/2}} \bar{P}^{3/2} \left\{1 + \frac{3e^2}{2} - \frac{\frac{3}{2}J_2 R_e^2}{\bar{P}^2} \frac{7 \cos^2 i_0 - 1}{4}\right\} \quad (34)$$

For small $e$, King-Hele's expressions for $\tau_\beta$ agree with those derived from Struble's, Sterne's, and Vinti's results.

**Blitzer,[10,11] Weisfeld,[10] Wheelon[10]**

Blitzer [Ref. 11, Eq. (14)] obtains the nodal period for circular orbits by a direct integration method and under the assumption that the radius $r_0$ remains constant during 1 rev. His result, for $e = 0$, with a change in notation and use of $\tau_0 = 2\pi/n_0$, becomes

$$\tau_\beta = \frac{2\pi}{(\mu)^{1/2}} \bar{r}_0^{3/2} \left\{1 - \frac{3}{2} J_2 \frac{R_e^2}{\bar{r}_0^2} \left(\frac{3 \cos^2 i_0 - 1}{2} + |\cos i_0| \sin^2 i_0\right)\right\} \quad (35)$$

Unfortunately, this result does not agree with the previous expressions for $\tau_\beta$. This seems to be due to Blitzer's assumption, in his integral, of a constant $r_0$ throughout a revolution, whereas $r_0$ undergoes variations to order $J$ which cannot be neglected in a first-order determination of $\tau_\beta$.

**Jaramillo[12]**

Jaramillo obtains the change in anomalistic period due to drag and oblateness for a circular orbit by a direct integration method. Jaramillo's average value for anomalistic period for one complete revolution (neglecting drag effects), as derived from his Eq. (64), with some change in notation, is

$$\tau_r = \frac{2\pi}{(\mu)^{1/2}} \bar{r}_0^{3/2} \left\{1 - \frac{9}{2} J_2 \frac{R_e^2}{\bar{r}_0^2} \sin^2 i_0 \cos 2\omega_0\right\} \quad (36)$$

while the value derived from his Eq. (I-151) becomes

$$\tau_r = \frac{2\pi}{(\mu)^{1/2}} \bar{r}_0^{3/2} \left\{1 - \frac{9}{4} J_2 \frac{R_e^2}{\bar{r}_0^2} \sin^2 i_0 \cos 2\omega_0\right\} \quad (37)$$

These expressions do not agree with each other, nor with the expressions for $\tau_r$ obtained by other authors. It is not clear whether the mean radius $r_0$ or $\bar{r}_0$ is used and how $\omega_0$, which Jaramillo defines as the longitude of the perigee point in the unperturbed orbit as measured from the ascending node, enters into the perturbed anomalistic period.

**Sturms[15]**

Sturms [Ref. 15, Eq. (23)], by use of the astronomical approach, obtains the nodal period for a "circular" earth satellite in terms of quantities of the unperturbed orbit:

$$\tau_\beta = \tau_0 \left\{1 - \frac{3}{2} J_2 \left(\frac{R_e}{a_0}\right)^2 \frac{11 \cos^2 i_0 - 5}{2}\right\} \quad (38)$$

Two peculiarities of Sturms' method are 1) that he defines circularity in terms of the instantaneous zero osculating eccentricity, i.e., the radius-time history of the "circular" orbit varies, depending on the initial central angle $\beta$, and 2) that he defines the perturbed orbit as one with the same angular momentum as the unperturbed orbit. If one uses this interpretation of the perturbed orbit, then the angular momentum per unit mass of the unperturbed circular satellite orbit is $(\mu a_0)^{1/2}$, and King-Hele's result [Ref. 5, Eq. (74)] agrees with Sturms' result, Eq. (38).

**Acceptable Formulations for $\tau_r$ and $\tau_\beta$**

Use of the *anomalistic period* has the advantage of it being independent of perigee location. Two serious disadvantages are that the perigee location changes with time, and that for near-circular and circular orbits the perigee is hard to define from observations and not necessarily the point of minimum geocentric radius.

Use of the *nodal period eliminates* these two disadvantages since the equatorial crossing is an easily defined point in elliptical or circular satellite orbits. Loosely speaking, the time between two successive crossings of any parallel of latitude in a given direction can be regarded as a nodal period. However, its use in satellite position prediction has the slight disadvantage that it is the anomalistic period plus the time $\Delta\tau$ it takes the satellite to traverse the angle $(d\omega/dt)\Delta\tau$. The value of $\Delta\tau$ depends on $\omega$ and $e$, and hence the nodal period experiences a second-order long-periodic oscillation (Ref. 14). Neglecting this small oscillation corresponds to the approximation of Eq. (12) in the present derivation of the nodal period. All presented expressions for $\tau_\beta$ give the mean value of the nodal period, i.e., they give the secular value of $\tau_\beta$ due to earth oblateness.

The problem, then, is to decide which forms of the anomalistic and nodal periods one should use. The forms for $\tau_\beta$ and $\tau_r$ given by Blitzer, Weisfeld, Wheelon, Jaramillo, and Sturms will be disregarded for the reasons stated. Expressions obtained by Kozai, Brouwer, Groves, by the authors from Struble, by Sterne, Vinti, and King-Hele for the anomalistic and nodal periods agree for a circular orbit. For near-circular orbits, with $e = O(J_2)$, the answers are expected to differ by less than one part in $10^5$. For larger eccentricities, the answers these authors obtained begin to diverge more and more

In the following, $\bar{a}$ is the mean semimajor axis of the perturbed orbit, $\bar{r}_0$ the perturbed orbital radius constant, $i_0$ the inclination constant, and $e_0$ the eccentricity constant, such that for a given set of $i_0$ and $\bar{r}_0$ or $i_0$, $\bar{a}$, $e_0$, the radius-time history of the orbit is fixed. The form

$$\tau_r = \frac{2\pi}{(\mu)^{1/2}} \bar{a}^{3/2} \left\{1 + \frac{3}{2} J_2 \frac{R_e^2}{\bar{a}^2(1 - e^2)^{3/2}} \left(\frac{3 \cos^2 i_0 - 1}{4}\right)\right\} \quad (39)$$

will be assumed as basic for the anomalistic period, since it has been obtained by several authors who used differing techniques.

As long as $e_0 = O(J)$, any of the forms given for the nodal period can be used to first order in oblateness. It is suggested that the forms derived by the authors from Struble

$$\tau_\beta = \frac{2\pi}{(\mu)^{1/2}} \bar{r}_0^{3/2} \left\{1 - \frac{3}{2} J_2 \left(\frac{R_e}{\bar{r}_0}\right)^2 \frac{7 \cos^2 i_0 - 1}{4}\right\} \quad (40)$$

or the one of Vinti, valid for small $e_0$,

$$\tau_\beta = \frac{2\pi}{(\mu)^{1/2}} \bar{a}^{3/2} \left\{1 - \frac{3}{2} J_2 \frac{R_e^2}{\bar{a}^2(1 - e_0^2)^2} \left(\frac{7 \cos^2 i_0 - 1}{4}\right)\right\} \quad (41)$$

be used, depending on whether Struble's method or the astronomical method is followed in the calculations.

The compatibility of the expressions, Eqs. (39–41), for $\tau_r$ and $\tau_\beta$ can be checked by computing time in orbit with them. Consider the limiting case of a circular orbit at arbitrary altitude and inclination. In a circular orbit $e_0 \equiv 0$, $\bar{a} \equiv \bar{r}_0$,

Table 1

| Orbit no. | $e_0$ | $\bar{a}$, naut miles | $\bar{P}$, naut miles | $i_0$, deg | $t_{orb}$ [from $\tau_r$ as given by Eq. (39)], sec | $t_{orb}$ [from $\tau_\beta$ as given by Eq. (31)], sec | $t_{orb}$ [from $\tau_\beta$ as given by Eq. (41)], sec |
|---|---|---|---|---|---|---|---|
| 1 | 0.1 | 4443.927 | 4399.48773 | 15 | 4,070,203 | 4,070,225 | 4,070,215 |
| 2 | 0.5 | 7443.927 | 5582.94525 | 45 | 34,754,300 | 34,755,020 | 34,754,660 |
| 3 | 0.1 | 4443.927 | 4399.48773 | 80 | 17,593,670 | 17,593,625 | 17,593,645 |
| 4 | 0.9 | 50000.000 | 9500.00000 | 90 | $2.628,940 \times 10^9$ | $2.628,781 \times 10^9$ | $2.628,860 \times 10^9$ |

and the time difference between the anomalistic and nodal periods can be related to the angle by which the perigee advances or regresses with respect to the ascending node. If one divides $(\tau_r - \tau_\beta)$ by $2\pi\mu^{-1/2}\bar{r}_0^{3/2}$ to obtain oblateness effects on the period in nondimensional form, then from Eqs. (39–41),

$$\frac{\tau_r - \tau_\beta}{2\pi\mu^{-1/2}\bar{r}_0^{3/2}} = \frac{3}{2}J_2\left(\frac{R_e}{r_0}\right)^2\left(\frac{5\cos^2 i_0 - 1}{2}\right) \tag{42}$$

Similarly, if the advance or regression of the perigee per nodal period is divided by 360, the oblateness effects can be obtained in nondimensional form from Eq. (17):

$$\frac{d\omega/dt}{360} = \tfrac{3}{2}J_2\left(\frac{R_e}{r_0}\right)^2\left(\frac{5\cos^2 i_0 - 1}{2}\right) \tag{43}$$

The right-hand sides of Eqs. (42) and (43) agree, thus indicating that the total time in a circular orbit, $t_{orb}$, is the same whether measured in terms of nodal or anomalistic revolutions.

For elliptic orbits, $e_0 \neq 0$, the comparison of $t_{orb}$ as measured in terms of nodal or anomalistic revolutions cannot be performed analytically, because the divisor in eccentricity is $(1 - e_0)^{3/2}$ in the expression for the anomalistic period and $(1 - e_0)^2$ for the nodal period. Some numerical checks were made on the compatibility of the expressions for $\tau_r$ and $\tau_\beta$ for four elliptic orbits with large $e_0$ and varying values of $i_0$. The number of revolutions it takes for the perigee to advance or regress 360° was established, thus determining a fixed number of anomalistic revolutions and a time in orbit from Eq. (39). At this moment, the number of nodal revolutions differs by one from that of the anomalistic revolutions, and a time in orbit can be calculated from Eq. (41). The difference between the time in orbit calculated by these methods is a measure of the compatibility of these two expressions. The result of these calculations, carried to ten places on the Friden calculator and by use of the constants $R_e = 3443.927$ naut miles, $\mu = 1.407645 \times 10^{16}$ ft$^3$/sec$^2$, and $J = 1.08228 \times 10^{-3}$, is shown in Table 1.

The time in orbit consistently closest to the one calculated from Eq. (39) for $\tau_r$ is obtained by use of Eq. (41) for $\tau_\beta$. Equation (31) for $\tau_\beta$ derived from Vinti and Sterne's expression and Eq. (26) for $\tau_\beta$ have been found to give a more divergent time in orbit from the one calculated by use of Eq. (39); in Sterne's case it is probably due to the particular form of the intermediary potential used. Hence Eq. (41) for $\tau\beta$ derived from Vinti's results is recommended for use together with Eq. (39) for $\tau_r$ obtained by Kozai, Groves, and Brouwer when first-order oblateness effects are considered. The expression for $\tau_\beta$ derived here, Eq. (40), is compatible with Eqs. (39) and (41) if $\bar{r}_0 \equiv \bar{P}$.

The numerical comparison in Table 1 indicates that the recommended expressions for $\tau_r$ and $\tau_\beta$ are also relatively compatible for large $e_0$, the difference being less than 1 part in $10^5$ for $e_0 = 0.1$, 0.5 and 3 parts in $10^4$ for $e_0 = 0.9$. A future paper will present a derivation of the nodal period from Struble's results for arbitrary $e_0$.

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